Matrices With Prescribed Nonprincipal Blocks and the Trace Function

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ABSTRACT

Let A and B be fixed matrices. We discuss the existence of matrices U and V such that $VAU^{-1} + UBV^{-1}$ is nonscalar and simultaneously has a prescribed trace.

In this paper we prove that, with a few exceptions, given two square matrices A and B over an arbitrary field, there exist invertible matrices U and V such that the matrix $VAU^{-1} + UBV^{-1}$ is nonscalar and has a prescribed trace. The exceptions occur when the field has two or three elements and then only for certain matrices A and B of order 2.

This question was suggested to me by G. N. de Oliveira when he found a flaw in the proof of one of his results about the existence of a matrix with prescribed nonprincipal blocks and characteristic polynomial [1]. In his proof he showed that a given pair of matrices was spectrally complete. However, from this fact it does not follow that a certain combination of matrices can have certain prescribed eigenvalues unless a trace condition is satisfied. Our present result fills the gap in Oliveira's proof, leaving only the exceptional cases, which are dealt with separately. In fact $VAU^{-1} + UBV^{-1}$ is one of the matrices of the pair we want to be spectrally complete, and so, as we can prescribe the trace for it, we can also arrange the trace condition referred to

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above. For the finite number of exceptional cases we checked de Oliveira's result directly.

As Oliveira's theorem was later used in a generalization by F. C. Silva [2], our present work ensures the validity of Silva's result.

THEOREM. *Let A and B be two nonzero $n \times n$ matrices over an arbitrary field $\mathbb{F}$. Let μ be an arbitrary element of $\mathbb{F}$. If $n \geqslant 3$, there exist invertible matrices U and V such that*:

(i) $VAU^{-1} + UBV^{-1}$ *is nonscalar.*
(ii) $\operatorname{tr}(VAU^{-1} + UBV^{-1}) = \mu$.

If $n = 2$, the result fails only in a finite number of cases.

Before proceeding to the proof, notice that we can replace one of the matrices by an equivalent one. Thus if the rank of A is r, we can assume that

$$A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix and the zeros represent blocks of appropriate size.

The proof is divided into several cases. In each case we just exhibit the matrices U and V, leaving the computations to the reader. Since U can always be taken to be I_n or a permutation matrix, in fact we only exhibit V.

Proof.

I. B has a nonzero off-diagonal element b_{ij}. We take as V the matrix obtained from I_n by adding its ith row multiplied by $(\operatorname{tr} A + \operatorname{tr} B - \mu)/b_{ij}$ to its jth row.

II. $B = \operatorname{diag}\{b_1, \dots, b_n\}$, $n \geqslant 3$.

(1) $1 \leqslant r \leqslant n-1$.

(α) *There is a nonzero b_i with $i \leqslant r$.* Let P be the permutation matrix obtained from I_n by interchanging the rows i and r. We can replace A and B by PAP^{-1} (this product leaves A unchanged) and PBP^{-1} respectively, which amounts to assuming in the initial problem that $b_r \neq 0$. We can now take

$$V = I_{r-1} \oplus \begin{bmatrix} 1 & -1 \\ -x & 1+x \end{bmatrix} \oplus I_{n-1-r},$$

where

$$x = \frac{\mu - \operatorname{tr} A - \operatorname{tr} B}{b_r}.$$

(β) $b_1 = \cdots = b_r = 0$.

(β_1) $r = 1$. Since $B \neq 0$, there is a b_i, with $i \geqslant 2$, different from zero. Performing a trick with a permutation matrix [as in case (α)], we may assume that $i = 2$.

(β_{11}) $\mu = \operatorname{tr} A + \operatorname{tr} B$. We take

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3}.$$

(β_{12}) $\mu \neq \operatorname{tr} A + \operatorname{tr} B$. We take

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^{-1} & -x^{-1} \\ 0 & -x^{-1} & x^{-1} + 1 \end{bmatrix} \oplus I_{n-3},$$

where

$$x = \frac{\mu - \operatorname{tr} A - \operatorname{tr} B}{b_2} \neq 0.$$

(β_2) $r \geqslant 2$.

(β_{21}) $\mu = \operatorname{tr} A + \operatorname{tr} B$. We take

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \oplus I_{n-2}.$$

(β_{22}) $\mu \neq \operatorname{tr} A + \operatorname{tr} B$. We take

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 1 + x \end{bmatrix} \oplus I_{n-2},$$

where

$$x = \mu - \operatorname{tr} A - \operatorname{tr} B \ (\neq 0).$$

(2) $r = n$.

(α) There are two different indices i and j, such that $b_i \neq -1$ and $b_j \neq 1$. Again, using a permutation matrix, we may assume that $i = 1$ and $j = 2$. We take

$$V = \begin{bmatrix} 1 & -x \\ -1 & 1+x \end{bmatrix} \oplus I_{n-2}, \qquad \text{where} \quad x = \frac{\mu - \operatorname{tr} A - \operatorname{tr} B}{b_1 + 1}.$$

(β) $b_1 = \cdots = b_n = -1$. We take

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & \mu & -\mu \end{bmatrix} \oplus I_{n-3}.$$

(γ) $b_1 = \cdots = b_n = 1$. We take

$$V = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & -a & a \end{bmatrix} \oplus I_{n-3},$$

$$\text{where} \quad a = \mu + 2 - 2(n-3).$$

(δ) $\operatorname{car}(\mathbb{F}) = 2$; *there is a b_i different from* 1, *the remaining b_j being* 1. We can assume that $b_1 \neq 1$. Since $\operatorname{car}(\mathbb{F}) = 2$, $\operatorname{tr}(A + B) = 1 + b_1 \neq 0$.

(δ_1) $\mu = 1 + b_1$. We take $V = I_n$.

(δ_2) $\mu \neq 1 + b_1$. We take

$$V = \begin{bmatrix} 1 & x \\ 1 & 1+x \end{bmatrix} \oplus I_{n-2}, \qquad \text{where} \quad x = \frac{\mu + 1 + b_1}{1 + b_1}.$$

Now we discuss the case $n = 2$. For this we suppose that A is already in the reduced form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

III. $B = \operatorname{diag}\{b_1, b_2\}$.

(1) $A = I_2$.

(α) $b_1 \neq -1$ *and* $b_2 \neq 1$. (The case $b_1 \neq 1$ and $b_2 \neq -1$ is the same, up to permutation.) We take

$$V = \begin{bmatrix} 1 & -x \\ -1 & 1+x \end{bmatrix}, \qquad \text{where} \quad x = \frac{\mu - \operatorname{tr} A - \operatorname{tr} B}{b_1 + 1}.$$

(β) $B = I_2$, $\mathbb{F} \neq \{0,1\}$, *and* $\mathbb{F} \neq \{0,1,-1\}$. There is a b in $\mathbb{F}$ such that $b \neq 0$, $b \neq 1$, and $b \neq -1$. We take

$$V = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}, \qquad \text{where} \quad a = \frac{\mu b}{b-1}.$$

(γ) $B = -I_2$, $\mathbb{F} \neq \{0,1\}$, *and* $\mathbb{F} \neq \{0,1,-1\}$. There is a b in $\mathbb{F}$ such that $b \neq 0$, $b \neq 1$ and $b \neq -1$. We take

$$V = \begin{bmatrix} a & 1 \\ -b & 0 \end{bmatrix}, \qquad \text{where} \quad a = \frac{\mu b}{b-1}.$$

(2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(α) $b_1 \neq 0$. We take

$$V = \begin{bmatrix} 1 & -1 \\ -x & 1+x \end{bmatrix}, \qquad \text{where} \quad x = \frac{\mu - \operatorname{tr} A - \operatorname{tr} B}{b_1}.$$

(β) $b_1 = 0$.

(β_1) $b_2 \neq 1$ *and* $b_2 \neq -1$. We take

$$V = \begin{bmatrix} 1+x & x \\ 1 & 1 \end{bmatrix}, \qquad \text{where} \quad x = \frac{\mu - 1 - b_2}{b_2 + 1}.$$

(β_2) $b_2 = 1$, $\mathbb{F} \neq \{0,1\}$, *and* $\mathbb{F} \neq \{0,1,-1\}$. There is a b in $\mathbb{F}$ such that $b \neq 0$, $b \neq 1$, and $b \neq -1$. We take

$$V = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}, \qquad \text{where} \quad a = \frac{\mu b}{b-1}.$$

(β_3) $b_2 = -1$, $\mathbb{F} \neq \{0,1\}$, *and* $\mathbb{F} \neq \{0,1,-1\}$. There is a b in $\mathbb{F}$ such that $b \neq 0$, $b \neq 1$, and $b \neq -1$. We take

$$V = \begin{bmatrix} a & 1 \\ -b & 0 \end{bmatrix}, \qquad \text{where} \quad a = \frac{\mu b}{b-1}.$$

In all the other cases the theorem fails. Before discussing them, notice firstly that, if

$$X = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$$

is a matrix over an arbitrary field and $\det X = 1$, then

$$X^{-1} = \begin{bmatrix} u_4 & -u_2 \\ -u_3 & u_1 \end{bmatrix},$$

and if $\det X = -1$,

$$X^{-1} = \begin{bmatrix} -u_4 & u_2 \\ u_3 & -u_1 \end{bmatrix}.$$

We discuss now the remaining cases.

(1) $\mathbb{F} = \{0, 1\}$.

(α) $A = I_2 = B$. We have

$$VAU^{-1} + UBV^{-1} = X + X^{-1}, \qquad \text{where} \qquad X = VU^{-1}.$$

Since U and V are nonsingular, the determinant of X must be 1. We see easily that $X + X^{-1}$ is always scalar.

(β) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. There is no difficulty in showing that $VAU^{-1} + UBV^{-1}$ is always scalar.

(2) $\mathbb{F} = \{0, 1, -1\}$.

(α) $A = I_2 = B$. Again we have $VAU^{-1} + UBV^{-1} = X + X^{-1}$. There are now two possibilities: either $\det X = 1$ or $\det X = -1$. In the first case $X + X^{-1}$ is scalar. In the second the trace of $X + X^{-1}$ is always zero.

(β) $A = I_2$, $B = -I_2$. We have $VAU^{-1} + UBV^{-1} = X - X^{-1}$. It is easy to show that $X - X^{-1}$ either is scalar or has trace zero.

(γ) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. In $VAU^{-1} + UBV^{-1}$ we have to consider four possibilities, as the determinants of U and V can be ± 1. We see that $VAU^{-1} + UBV^{-1}$ is either scalar or a zero-trace matrix.

(δ) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. As in case ($\gamma$), we see that $VAU^{-1} + UBV^{-1}$ is either scalar or a zero-trace matrix. ∎

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